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Random walks on Ramanujan complexes and digraphs

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Abstract. The cutoff phenomenon was recently confirmed for random walks on Ramanujan graphs by the first author and Peres. In this work, we obtain analogs in higher dimensions, for random walk operators on any Ramanujan complex associated with a simple group G over a local field F . We show that if T is any k -regular G -equivariant operator on the Bruhat–Tits building with a simple combinatorial property (collision-free), the associated random walk on the n -vertex Ramanujan complex has cutoff at time $\log_k n$. The high-dimensional case, unlike that of graphs, requires tools from non-commutative harmonic analysis and the infinite-dimensional representation theory of G . Via these, we show that operators T as above on Ramanujan complexes give rise to Ramanujan digraphs with a special property (r -normal), implying cutoff. Applications include geodesic flow operators, geometric implications, and a confirmation of the Riemann Hypothesis for the associated zeta functions over every group G , previously known for groups of type A_n and \tilde{C}_2 .

Keywords. Ramanujan complexes, high dimensional expanders, mixing time of random walk, cutoff phenomenon

1. Introduction

A finite connected k -regular graph \mathcal{G} is *Ramanujan* (as defined in [34]) if every eigenvalue $\lambda \in \mathbb{R}$ of its adjacency matrix satisfies either $|\lambda| \leq 2\sqrt{k-1}$ or $|\lambda| = k$. In [31], the cutoff phenomenon was verified for simple random walk on such graphs: the total variation distance of the walk from its stationary distribution exhibits a sharp transition at time $\frac{k}{k-2} \log_{k-1} n$, whereby it drops from near the maximum to near zero along an interval of $o(\log n)$ steps. One of the consequences of that analysis was that the typical distance between vertices in \mathcal{G} is $(1 + o(1)) \log_{k-1} n$. Our goal is to establish analogous results in higher dimensions, for a broad class of random walk operators on Ramanujan complexes, as defined next.

Let \mathbf{G} be a simple algebraic group of rank d , defined over a non-Archimedean local field F , with residue field of order q . Let $G = \mathbf{G}(F)$, and let $\mathcal{B} = \mathcal{B}(G)$ be the associated

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Bruhat–Tits building, which is a d -dimensional contractible simplicial complex. Fix a d -dimensional cell σ_0 of \mathcal{B} (a “chamber”) and let \mathcal{I} denote the Iwahori subgroup of G , which is the (pointwise) stabilizer of σ_0 in G , uniquely determined up to conjugation (as G acts transitively on the chambers). Let Γ be a torsion-free cocompact discrete subgroup of G . The quotient $X = \Gamma \backslash \mathcal{B}$ is a finite d -dimensional simplicial complex, and we say it is a *Ramanujan complex*¹ if

$$\begin{aligned} &\text{every irreducible } \mathcal{I}\text{-spherical infinite-dimensional} \\ &\text{\mathcal{G}\text{-subrepresentation of } } L^2(\Gamma \backslash G) \text{ is tempered,} \end{aligned} \quad (1.1)$$

where a representation is *tempered* if it is weakly contained in $L^2(G)$, and it is *\mathcal{I} -spherical* if it contains an \mathcal{I} -fixed vector.

A k -branching operator T on a subset of cells $\mathcal{C} \subset \mathcal{B}$ is a map from \mathcal{C} to $\binom{\mathcal{C}}{k}$, the k -element subsets of \mathcal{C} ; one can identify T with the adjacency matrix of $\mathcal{Y}_{T, \mathcal{C}}$, the k -out-regular digraph on the vertex set \mathcal{C} , where (x, y) is an edge iff $y \in T(x)$. When $\mathcal{Y}_{T, \mathcal{C}}$ is also k -in-regular, we say that T is a k -regular branching operator. We focus on \mathcal{C} that is G -invariant, and furthermore T that is G -equivariant (that is, $T(gx) = gT(x)$ for all $x \in \mathcal{C}$ and $g \in G$). In this case, the k -branching operator T on $\mathcal{C} \subset \mathcal{B}$ induces such an operator on the finite subset of cells $\mathfrak{X} = \Gamma \backslash \mathcal{C} \subset X$. We will study the mixing time of the random walk on \mathfrak{X} associated with T .

Let $\|\mu - \nu\|_{\text{TV}} = \sup_A [\mu(A) - \nu(A)] = \frac{1}{2} \|\mu - \nu\|_{L^1}$ denote the total-variation distance on a countable state space. The L^1 mixing time of a finite ergodic Markov chain with transition kernel P , stationary distribution π and a worst initial state, is

$$t_{\text{MIX}}(\varepsilon) = \min\{t : D_{\text{TV}}(t) \leq \varepsilon\} \quad \text{where} \quad D_{\text{TV}}(t) = \max_x \|P^t(x, \cdot) - \pi\|_{\text{TV}}.$$

A sequence of finite ergodic Markov chains is said to exhibit the *cutoff phenomenon* if its total-variation distance from stationarity drops abruptly, over a period of time referred to as the *cutoff window*, from near 1 to near 0; that is, there is cutoff iff $t_{\text{MIX}}(\varepsilon) = (1 + o(1))t_{\text{MIX}}(\varepsilon')$ for any fixed $0 < \varepsilon, \varepsilon' < 1$. This phenomenon, discovered by Aldous and Diaconis in the early 1980s (see [1, 2, 9, 11]), was verified only in relatively few cases, though believed to be widespread (cf. [29, §18]).

Peres conjectured in 2004 that simple random walk (SRW) on every family of transitive expanders exhibits cutoff, yet there was not a single example of such a family prior to the result on Ramanujan graphs in [31]. The main step there was to show, via spectral analysis, that the *nonbacktracking* random walk (NBRW)—that does not traverse the same edge twice in a row—has cutoff at time $\log_{k-1} n$ on any k -regular n -vertex Ramanujan graph. As it turns out, spectral analysis of the SRW does not directly establish L^1 cutoff, as the SRW has L^1 and L^2 cutoffs at different times. On the other hand, for the NBRW—which may be viewed as the version of SRW that, on the universal cover of the graph (the k -regular tree), never creates cycles—the locations of the L^1 and L^2 cutoffs do coincide.

¹ The definition given here slightly differs from the one first given in [36]; see the discussion in §4.

The analog of this phenomenon in higher dimensions would say that if the random walk operator T satisfies a combinatorial property on the building \mathcal{B} (generalizing the notion of avoiding cycles on the universal cover) then it should exhibit cutoff on any Ramanujan complex $X = \Gamma \backslash \mathcal{B}$ as defined above. For the NBRW, the digraph $\mathcal{Y}_{T, \mathfrak{C}}$ associated with the corresponding operator T on the universal cover is nothing more than the infinite directed (away from the root) k -regular tree. The combinatorial criterion defined next broadens the class of permissible digraphs $\mathcal{Y}_{T, \mathfrak{C}}$ supporting cutoff for T on any Ramanujan complex X .

Call a k -branching operator T *collision-free* if its associated digraph $\mathcal{Y}_{T, \mathfrak{C}}$ has at most one (directed) path from x to y for every $x, y \in \mathfrak{C}$. Our main result is that this combinatorial criterion implies cutoff for all Ramanujan complexes.

Theorem 1. *Let \mathcal{B} be the Bruhat–Tits building associated with a simple algebraic group G of rank $d \geq 1$ over a locally compact non-Archimedean field. Let T be a G -equivariant and collision-free k -regular branching operator on a subset of cells $\mathfrak{C} \subseteq \mathcal{B}$. Let $X = \Gamma \backslash \mathcal{B}$ be any Ramanujan complex as defined in (1.1), let $\mathfrak{X} = \Gamma \backslash \mathfrak{C} \subseteq X$, and set $n = |\mathfrak{X}|$. Let $\rho(x, y) = \min\{\ell : y \in T^\ell(x)\}$ for $x, y \in \mathfrak{X}$ (shortest-path metric in $\mathcal{Y}_{T, \mathfrak{X}}$), and assume without loss of generality that $\max_{x, y} \rho(x, y) < \infty$.*

- (i) *There exist $c_G, M_G > 0$, depending only on G , such that the following holds. If Y_t is the random walk associated with T on \mathfrak{X} , modified in its first step to perform $U_0 \sim \text{Uniform}(\{1, \dots, M_G\})$ steps instead of one, then for every fixed $0 < \delta < 1$ and n large enough,*

$$|t_{\text{MIX}}(\delta) - \log_k n| \leq c_G \log_k \log n.$$

In particular, Y_t exhibits worst-case total-variation cutoff at time $\log_k n$.



- (ii) *There exists $c_G > 0$, depending only on G , such that for every $x \in \mathfrak{X}$,*

$$\#\{y \in \mathfrak{X} : |\rho(x, y) - \log_k n| > c_G \log_k \log n\} = o(n).$$

Representation theory has been used to show cutoff for random walks on groups, notably in the pioneering work of Diaconis and Shahshahani [11] (for more on this, see [10], the monograph by Diaconis [8] and survey by Saloff-Coste [37]); however, this is perhaps the first case where the infinite-dimensional representation theory of locally compact groups is used to study such problems.

In §5.2, we show that operators such as T above, acting on the chambers of the building, are parameterized as $T = T_{w_0}$, where w_0 is an element of the affine Weyl group \mathcal{W} of G . In Theorem 5.6, we give a simple necessary and sufficient condition on $w_0 \in \mathcal{W}$ for T_{w_0} to be collision-free. Moreover, we show that various geodesic flow operators on cells of all dimensions (see Definition 5.1) are collision-free.

Example 1.1. For $d = 2$, one has j -dimensional geodesic flow operators ($j = 1, 2$) on tripartite Ramanujan complexes, with the color of a vertex x denoted by $\text{col}(x)$:

j	k -branching geodesic flow operator T	k	cutoff location
1	$\mathfrak{X} = \{(x, y) \in X : \text{col}(y) \equiv \text{col}(x) + 1\}$ $T(x, y) = \{(y, z) \in \mathfrak{X} : \{x, y, z\} \notin X\}$		q^2 $\frac{1}{2} \log_q n$
2	$\mathfrak{X} = \left\{ (x, y, z) \in X : \begin{array}{l} \text{col}(y) \equiv \text{col}(x) + 1 \\ \text{col}(z) \equiv \text{col}(y) + 1 \end{array} \right\}$ $T(x, y, z) = \{(y, z, w) \in \mathfrak{X}, w \neq x\}$		q $\log_q n$

See §5.1 for the general case, where the j -dimensional geodesic flow ($1 \leq j \leq d$) has cutoff at time $(d + 1 - j)^{-1} \log_q n$.

The cutoff locations and shortest-path lengths in Theorem 1 are optimal among operators with maximal out-degree k : the walk cannot reach a linear number of vertices, let alone mix, by time $(1 - \varepsilon) \log_k n$. This shows the extremality of such walks on Ramanujan complexes vs. any k -out-regular walk on an n -element set.

Let us stress another interesting point: in recent years, Ramanujan complexes have been instrumental in tackling several open problems (see, e.g., [13, 14, 17, 26]). In all these applications, one could settle for less than the Ramanujan property, and in fact, a quantitative version of Kazhdan's Property (T) would suffice (see for instance [12]). In this work, on the other hand, the Ramanujan property is crucial for the analysis of the walk operators on the complexes to be sharp.

An important tool in this work is the directed analog of Ramanujan graphs. We define a finite k -out-regular digraph \mathcal{Y} to be Ramanujan if the following holds:

$$\begin{aligned} &\text{every eigenvalue } \lambda \in \mathbb{C} \text{ of the adjacency matrix} \\ &\text{of } \mathcal{Y} \text{ satisfies either } |\lambda| \leq \sqrt{k} \text{ or } |\lambda| = k. \end{aligned} \quad (1.2)$$

In these terms, the fundamental result of Hashimoto [19], which characterized the spectrum of the NBRW operator on a graph, shows that if \mathcal{G} is a k -regular undirected Ramanujan graph, then the digraph $\mathcal{Y}_{T, \mathcal{G}}$ corresponding to the NBRW operator T on \mathcal{G} is a Ramanujan digraph. This spectral property was one of two key ingredients in the mixing time analysis of the NBRW in [31]; the other—an algebraic property (established in that work for any graph)—was that the adjacency matrix of $\mathcal{Y}_{T, \mathcal{G}}$ is unitarily similar to a block-diagonal matrix with blocks of size at most 2 and bounded entries.

Generalizing this notion, we call a matrix r -normal if it is unitarily similar to a block-diagonal matrix where each of the blocks has size at most r , and say that an operator T on $\mathfrak{X} \subset X$ is r -normal if the adjacency matrix of $\mathcal{Y}_{T, \mathfrak{X}}$ is r -normal. We further call T irreducible if $\mathcal{Y}_{T, \mathfrak{X}}$ is strongly connected, and then say its period is the gcd of all directed cycles in $\mathcal{Y}_{T, \mathfrak{X}}$.

To prove the main theorem, we show that under its hypotheses, T is r -normal for some $r = r(G)$; furthermore, its corresponding digraph $\mathcal{Y}_{T, \mathfrak{X}}$ is Ramanujan. Proposition 3.1, which is of independent interest, establishes total-variation cutoff for such digraphs by following the argument of [31, Theorem 3.5], extending to general r what was proved

there for $r = 2$. As shown in that work for graphs (see [31, Corollaries 2 and 3]; also see [38, Theorem 1.3] for an alternative proof, discovered independently, of the result on typical distances in Ramanujan graphs), this allows one to estimate key geometric features of Ramanujan digraphs:

Theorem 2. *For fixed $k, M, r \geq 1$, let \mathcal{Y} denote a k -regular strongly connected Ramanujan digraph on n vertices, and suppose that its adjacency matrix $A_{\mathcal{Y}}$ has period M and is r -normal. For every $x \in V(\mathcal{Y})$, there is a directed path of length at most $\log_k n + (2r - 1 + o(1)) \log_k \log n$ from x to y , and such a path from y to x , for all but $o(n)$ of the vertices $y \in V(\mathcal{Y})$. In particular, $\text{diam}(\mathcal{Y}) \leq (2 + o(1)) \log_k n$.*

Theorem 1 is deduced from Proposition 3.1 and Theorem 2 using the following result, which reduces branching operators on Ramanujan complexes to digraphs.

Theorem 3. *Let T denote a collision-free, G -equivariant, k -regular branching operator on $\mathcal{C} \subseteq \mathcal{B}$, let $X = \Gamma \backslash \mathcal{B}$ be a Ramanujan complex as defined in (1.1), and let $\mathfrak{X} = \Gamma \backslash \mathcal{C}$. Then the digraph $\mathcal{Y}_{T, \mathfrak{X}}$ is Ramanujan and r -normal for $r = r(G)$.*

Remark 1.2. Passing from Ramanujan complexes to Ramanujan digraphs is by no means a tautology; e.g., neither the graphs nor digraphs associated with SRW on the 1-skeleton/chambers of a Ramanujan complex are Ramanujan (see [36]).

Remark 1.3. We stress that r cannot be bounded as a function of k . Such is the case for the geodesic flow operators mentioned in Example 1.1 in general dimension (see Propositions 5.3 and 5.4).

The reduction step from Ramanujan complexes to Ramanujan digraphs uses representation theory and harmonic analysis of G : If λ is an eigenvalue of the digraph induced by T on $\mathfrak{X} = \Gamma \backslash \mathcal{C}$, with an eigenfunction f , then f induces a matrix coefficient φ_f on G , which is a λ -eigenfunction of T on \mathcal{C} . If $X = \Gamma \backslash \mathcal{B}$ is Ramanujan, then φ_f is a matrix coefficient of a tempered representation. A general result on semisimple groups in [6] ensures that φ_f is in $L^{2+\varepsilon}$ for all $\varepsilon > 0$. We then show that an $L^{2+\varepsilon}$ eigenvalue λ of a collision-free k -regular operator T satisfies $|\lambda| \leq \sqrt{k}$, which implies that $\mathcal{Y}_{T, \mathfrak{X}}$ is a Ramanujan digraph. Using the fact that the space $L^2(\Gamma \backslash G/P)$, when P is a parahoric subgroup, decomposes as a direct sum of irreducible representations of the Hecke–Iwahori algebra, whose irreducible representations are all of bounded dimension, we deduce that it is also r -normal for some r (see §4).

Finally, our analysis of random walk operators also implies a result on the zeta functions of Ramanujan complexes. Recall (see §6 for further details) that Ihara [20] and Hashimoto [19] associated zeta functions with k -regular (di)graphs, and showed that the graph is Ramanujan iff the zeta function satisfies the Riemann hypothesis (R.H.). The zeta function of a digraph \mathcal{Y} is $Z_{\mathcal{Y}}(u) = \prod_{[\gamma]} \frac{1}{1 - u^{\ell(\gamma)}}$, where $[\gamma]$ runs over equivalence classes of primitive directed cycles of \mathcal{Y} (see §6). We say that it satisfies the R.H. if $Z_{\mathcal{Y}}(k^{-s}) = \infty$ implies $|s| = 1$ or $\Re(s) \leq 1/2$.

Following W. Li, various authors have defined zeta functions associated with Ramanujan complexes, and verified the R.H. for them over groups of type \tilde{A}_n and \tilde{C}_2

(see [15, 22–25]). These can all be described in our notation as $Z_{\mathcal{Y}_{T,\mathfrak{X}}}(u)$ for suitable collision-free branching operators T on \mathcal{B} (see §6). As a corollary of Theorem 3, we thus have a far reaching generalization of all of these results:

Corollary 4. *Let X be a Ramanujan complex as in (1.1), and let $Z_{\mathcal{Y}_{T,\mathfrak{X}}}(u)$ be the zeta function associated with a G -equivariant, collision-free, k -regular branching operator T on $\mathcal{B} = \mathcal{B}(G)$. Then $Z_{\mathcal{Y}_{T,\mathfrak{X}}}(u)$ satisfies the Riemann Hypothesis, that is, if $Z_{\mathcal{Y}_{T,\mathfrak{X}}}(u)$ has a pole at k^{-s} then $|s| = 1$ or $\Re(s) \leq 1/2$.*

We should mention that Kang [22] (and the other references mentioned above) obtained more detailed information about the location of the poles, by carefully examining the representations appearing in these cases.

Note that Corollary 4 is only part of the story; as asked explicitly by Kang [22], one would like to have the converse, i.e., that the R.H. implies the Ramanujan property. This (and much more) has been established recently by Kamber [21]. The reader is referred to his paper for a wealth of interesting related results.

2. Spectrum of branching operators

Let X be a countable set and, as usual, let $L^p = L^p(X)$ for $1 \leq p \leq \infty$ denote its complex L^p -space. Every k -branching operator T on X gives rise to a bounded linear map $A = A_{\mathcal{Y}_{T,X}} : L^p(X) \rightarrow L^p(X)$ for every p , where $A_{\mathcal{Y}_{T,X}}$ corresponds to the adjacency matrix of $\mathcal{Y}_{T,X}$, the digraph associated with T ; namely,

$$(Af)(x) = \sum_{y \in T(x)} f(y).$$

We are mostly interested in the spectrum of A on L^2 , but our analysis will also consider it as acting on L^p for $p \neq 2$.

Definition 2.1. Let A be the map associated with a k -branching operator T .

(1) The *spectrum* of A (on L^2) is

$$\text{spec}(A) = \{\lambda \in \mathbb{C} : (\lambda I - A)|_{L^2} \text{ does not have a bounded inverse}\},$$

and the *spectral radius* of A is $\rho(A) = \max\{|\lambda| : \lambda \in \text{spec}(A)\}$.

(2) The *p -point spectrum* of A is

$$\text{Pspec}_p(A) = \{\lambda \in \mathbb{C} : \exists f \in L^p \text{ such that } Af = \lambda f\}.$$

(3) The *2^+ -point spectrum* of A is

$$\text{Pspec}_{2^+}(A) = \left\{ \lambda \in \mathbb{C} : \exists f \in \bigcap_{\varepsilon > 0} L^{2+\varepsilon} \text{ such that } Af = \lambda f \right\}.$$

(4) The *approximate point spectrum* of A is

$$\text{APspec}(A) = \left\{ \lambda \in \mathbb{C} : \exists f_n \in L^2 \text{ such that } \frac{\|Af_n - \lambda f_n\|}{\|f_n\|} \rightarrow 0 \right\}.$$

For finite X , all these definitions are equivalent, but otherwise they may differ. In particular, while $\text{APspec}(A) = \text{spec}(A)$ holds for a *normal* operator A , this may not hold in general; in this work we will (mainly) consider operators that are not normal, so this distinction is significant for us. We start with a trivial bound:

Proposition 2.2. *If $T : X \rightarrow 2^X$ is a branching operator with in-degrees and out-degrees bounded by k , then $\rho(A_{\mathcal{Y}_T}) \leq \|A_{\mathcal{Y}_T}\|_2 \leq k$.*

Proof. The bound $\rho(A_{\mathcal{Y}_T}) \leq \|A_{\mathcal{Y}_T}\|_2$ holds for any bounded operator, and if $f \in L^2(X)$ with $\|f\|_2 = 1$ then

$$\|A_{\mathcal{Y}_T} f\|_2^2 = \sum_{x \in X} \left| \sum_{y \in T(x)} f(y) \right|^2 \leq k \sum_{x \in X} \sum_{y \in T(x)} |f(y)|^2 \leq k^2 \sum_{y \in X} |f(y)|^2 = k^2,$$

where the first inequality is by Cauchy–Schwarz. \square

Clearly, if T is a k -branching collision-free operator, then the associated digraph \mathcal{Y}_T must be infinite. Such a digraph looks like a tree from the point of view of each $y_0 \in \mathcal{Y}$. Still we should warn that, once the orientation is ignored, it may have non-trivial closed paths. Moreover, its spectrum can be very different from that of a tree—see Example 2.4 for an “amenable” collision-free digraph. Still, the 2^+ -point spectrum of such a graph is very limited.

Proposition 2.3. *Let $\mathcal{Y} = \mathcal{Y}_T$ for a k -branching collision-free operator T . Then every $\lambda \in \text{PSpec}_{2^+}(A_{\mathcal{Y}})$ satisfies $|\lambda| \leq \sqrt{k}$.*

Proof. Let $f \in L^\infty(\mathcal{Y})$ be an eigenfunction of $A_{\mathcal{Y}}$ with eigenvalue λ . Scale f so that $f(v) = 1$ for some v , and fix $\varepsilon > 0$. Let us show that $T^j(v)$, the j -th level in the subtree of \mathcal{Y} emanating from v , satisfies

$$\sum_{w \in T^j(v)} |f(w)|^{2+\varepsilon} \geq \left| \frac{\lambda^{2+\varepsilon}}{k^{1+\varepsilon}} \right|^j.$$

This indeed holds for $j = 0$. Notice that if w, w' are different vertices in $T^j(v)$, then $T(w)$ and $T(w')$ are disjoint, since \mathcal{Y} is collision-free. This allows us to proceed by induction:

$$\sum_{w \in T^{j+1}(v)} |f(w)|^{2+\varepsilon} = \sum_{u \in T^j(v)} \sum_{w \in T(u)} |f(w)|^{2+\varepsilon} \geq \frac{1}{k^{1+\varepsilon}} \sum_{u \in T^j(v)} \left(\sum_{w \in T(u)} |f(w)| \right)^{2+\varepsilon},$$

where the last inequality is a consequence of Hölder’s inequality, since for any Hölder conjugates p, q one has $\sum_{i=1}^k |a_i|^p \geq k^{-p/q} (\sum_{i=1}^k |a_i|)^p$. The right-hand side of the last display is, in turn, at least

$$\frac{1}{k^{1+\varepsilon}} \sum_{u \in T^j(v)} |\lambda f(u)|^{2+\varepsilon} = \frac{|\lambda|^{2+\varepsilon}}{k^{1+\varepsilon}} \sum_{u \in T^j(v)} |f(u)|^{2+\varepsilon} \geq \left| \frac{\lambda^{2+\varepsilon}}{k^{1+\varepsilon}} \right|^{j+1}.$$

Since \mathcal{Y} is collision-free, the levels $T^j(v)$ are disjoint for different j 's, so that

$$\|f\|_{2+\varepsilon}^{2+\varepsilon} \geq \sum_{j=0}^{\infty} \sum_{w \in T^j(v)} |f(w)|^{2+\varepsilon} \geq \sum_{j=0}^{\infty} \left| \frac{\lambda^{2+\varepsilon}}{k^{1+\varepsilon}} \right|^j,$$

which is finite iff $|\lambda| < k^{\frac{1+\varepsilon}{2+\varepsilon}}$. Thus if, $f \in \bigcap_{\varepsilon>0} L^{2+\varepsilon}(\mathcal{Y})$, then $|\lambda| \leq \sqrt{k}$. \square

However, it is not necessarily the case that an arbitrary λ in the spectrum of $A_{\mathcal{Y}}$ on $L^2(\mathcal{Y})$ has $|\lambda| \leq \sqrt{k}$, and so \mathcal{Y} is not necessarily Ramanujan; indeed, the following example shows that the situation for the full spectrum of a collision-free digraph can be drastically different from the 2^+ -point spectrum.

Example 2.4 (Amenable collision-free digraph). Let $V = \mathbb{N} \times (\mathbb{Z}/k\mathbb{Z}[x])$ with each $(n, p(x))$ connected to the k vertices $(n+1, p(x) + a \cdot x^n)$ with $a \in \mathbb{Z}/k\mathbb{Z}$. This digraph is k -out-regular, and for any $m \in \mathbb{N}$ one can take

$$f = \mathbb{1}_{\{(0, p(x)): \deg p(x) < m\}},$$

for which $T^m f = k^m \cdot \mathbb{1}_{\{(m, p(x)): \deg p(x) < m\}}$, implying

$$\|T^m\| \geq \frac{\|T^m f\|}{\|f\|} = \frac{k^{3m/2}}{k^{m/2}} = k^m.$$

This gives $\rho(T) = k$, since $\rho(T) \leq k$ by Proposition 2.2 and in general one has $\rho(T) = \lim_{m \rightarrow \infty} \sqrt[m]{\|T^m\|}$. In fact, k is even in the approximate point spectrum of T , since, for $f_m = \mathbb{1}_{\{(j, p(x)): j < m, \deg p(x) < m\}}$,

$$T(f_m) = k \cdot \mathbb{1}_{\{(j, p(x)): 1 \leq j \leq m, \deg p(x) < m\}},$$

and hence $\|(T - k)f_m\|/\|f_m\| \xrightarrow{m \rightarrow \infty} 0$.

In §4 we will show that in the cases we focus on—affine buildings associated with simple groups—spectral behaviors as in the example above cannot occur.

3. Random walks on Ramanujan digraphs

The goal of this section is to establish the following cutoff result for Ramanujan digraphs, generalizing [31, Theorem 3.5] from $r = 2$ to any fixed r .

In what follows, say that a matrix A indexed by $V \times V$ is r -normal with respect to a distribution π on V if there exists a block-diagonal matrix Λ , whose blocks are each of size at most r , and a basis $\{w_i\}_{i \in V}$ of \mathbb{C}^V , such that $AW = W\Lambda$ for the matrix W which has the w_i 's as its columns, and the w_i 's form an orthonormal system with respect to the inner product $\langle f, g \rangle_{L^2(\pi)} := \sum_{x \in V} f(x) \overline{g(x)} \pi(x)$. We will look at distributions π where the ratio between the maximum and minimum probabilities is at most $\log^\alpha |V|$ for some $\alpha \geq 0$ (in our main result, π is uniform, i.e., $\alpha = 0$).

Proposition 3.1. Fix $k, b, M, r \geq 1$, $\alpha \geq 0$, and let \mathcal{Y} be a k -out-regular n -vertex strongly connected digraph, with all in-degrees at most b , and a period dividing M . Let Y_t be SRW on \mathcal{Y} , modified to make $U_0 \sim \text{Uniform}(\{1, \dots, M\})$ steps in its first move, let π be its stationary distribution, and suppose $\|\pi\|_\infty / \|\pi^{-1}\|_\infty \leq \log^\alpha n$. If \mathcal{Y} is Ramanujan and its adjacency matrix $A_{\mathcal{Y}}$ is r -normal with respect to π , then

$$|t_{\text{MIX}}(\delta) - \log_k n| \leq (2r - 1 + \alpha + o(1)) \log_k \log n \quad \text{for every fixed } 0 < \delta < 1.$$

In particular, Y_t exhibits worst-case total-variation cutoff at time $\log_k n$.

Proof. The lower bound on $t_{\text{MIX}}(\delta)$ will follow immediately from the following elementary lower bound on the mixing time of random walk (cf. [32, Claim 4.8]).

Observation 3.2. Let (Y_t) be simple random walk on a digraph \mathcal{Y} with n vertices and maximum out-degree $k \geq 2$, and suppose that its stationary distribution π satisfies $\|\pi\|_\infty \leq (\log n)^\alpha / n$ for some $\alpha > 0$. Then

$$D_{\text{TV}}(T) \geq 1 - 1/\log n \quad \text{at} \quad T = \lfloor \log_k n - (\alpha + 1) \log_k \log n \rfloor. \quad (3.1)$$

Indeed, for any initial vertex $x_0 \in V(\mathcal{Y})$, if S is the set of vertices supporting the distribution of Y_T then $|S| \leq k^T \leq n/(\log n)^{\alpha+1}$ gives $\pi(S) \leq \|\pi\|_\infty |S| \leq 1/\log n$.

For a matching upper bound, first consider the aperiodic case $M = 1$. Since \mathcal{Y} is strongly connected, this corresponds to the case where the only eigenvalue λ with $|\lambda| = k$ is k itself, which appears with multiplicity 1. We will show that if $v_t^{x_0} = \mathbb{P}_{x_0}(Y_t \in \cdot)$ is the distribution of (Y_t) starting from $X_0 = x_0$, then

$$\max_{x_0 \in V(\mathcal{Y})} \|v_{t_\star}^{x_0} / \pi - \mathbb{1}\|_{L^2(\pi)}^2 \leq \left(\frac{r^3 b^{r-1}}{(\log k)^{2r-2}} + o(1) \right) \frac{1}{\log n} \quad (3.2)$$

at

$$t_\star = \lceil \log_k n + (2r - 1 + \alpha) \log_k \log n \rceil. \quad (3.3)$$

In particular, this will imply that $\max_{x_0} \|v_{t_\star}^{x_0} / \pi - \mathbb{1}\|_{L^1(\pi)} = O(1/\sqrt{\log n})$, and so $t_{\text{MIX}}(\delta) \leq t_\star$ for every fixed $\delta > 0$ and large enough n , concluding the case $M = 1$.

Let $A_{\mathcal{Y}} W = W \Lambda$ be the matrices per the r -normal decomposition of A . That is, $\Lambda = \text{diag}(B_1, \dots, B_m)$, where B_l ($l = 1, \dots, m$) is an $s_l \times s_l$ upper triangular matrix with $s_l \leq r$, and W is a matrix whose columns, denoted $(w_{l,i})_{1 \leq l \leq m, 1 \leq i \leq s_l}$, form an orthonormal system with respect to $\langle \cdot, \cdot \rangle_{L^2(\pi)}$.

Since $k^{-1} A_{\mathcal{Y}}$ has π as a left eigenvector with eigenvalue 1 (by definition of π), it follows that every right eigenvector v of $A_{\mathcal{Y}}$ with eigenvalue $\lambda \neq k$ is orthogonal to π with respect to the standard complex inner product. Equivalently, every such vector v satisfies $\langle v, \mathbb{1} \rangle_{L^2(\pi)} = 0$, and in particular, since the eigenvalue $\lambda = k$ is simple, we can assume that $w_{1,1} = \mathbb{1}$, corresponding to $s_1 = 1$ and $B_1 = (k)$ in Λ , while each B_l for $l > 1$ has diagonal entries $(\lambda_{l,1}, \dots, \lambda_{l,s_l})$ satisfying $|\lambda_{l,i}| \leq \sqrt{k}$ for all $1 \leq i \leq s_l$.

We claim that, with this notation,

$$\max_{i,j} |B_l(i, j)| \leq \sqrt{k b} \quad \text{and} \quad \|B_l\|_F \leq \sqrt{r k b}, \quad (3.4)$$

where $\|\cdot\|_F$ is the Frobenius norm. Indeed,

$$\|A_Y\|_2^2 = \rho(A_Y A_Y^*) \leq \|A_Y A_Y^*\|_\infty \leq \|A_Y\|_\infty \|A_Y^*\|_\infty = \|A_Y\|_\infty \|A_Y\|_1 \leq kb.$$

In particular, for every $1 \leq j \leq s_l$, the left-hand side of the identity

$$A_Y w_{l,j} = \sum_{i \leq j} B_l(i, j) w_{l,i}$$

has $\|A_Y w_{l,j}\|_2^2 \leq \|A_Y\|_2^2 \leq kb$, and so applying Parseval's identity to its right-hand side yields

$$kb \geq \max_{1 \leq j \leq s_l} \left\| \sum_{i \leq j} B_l(i, j) w_{l,i} \right\|_2^2 = \max_{1 \leq j \leq s_l} \sum_{i \leq j} |B_l(i, j)|^2,$$

thus implying (3.4).

For $l \geq 2$, every B_l may be written as $D_l + N_l$, where $D_l = \text{diag}(\lambda_{l,1}, \dots, \lambda_{l,s_l})$ and N_l is a nilpotent matrix of index at most r . We can therefore infer from (3.4) and the bound $\max_j |\lambda_{l,j}| \leq \sqrt{k}$ that, for all $l \geq 2$ and every t ,

$$\max_{i,j} |B_l^t(i, j)| \leq \sum_{h=0}^{r-1} \binom{t}{h} k^{(t-h)/2} (kb)^{h/2} \leq r(t\sqrt{b})^{r-1} k^{t/2}. \quad (3.5)$$

This bound will allow us to bound the L^2 -distance from $v_t^{x_0}$ to equilibrium. Recall that $v_t^x(y) = k^{-t} A_Y^t(x, y)$; as $A_Y W = W \Lambda$, it follows that for every l, j, t ,

$$A_Y^t w_{l,j} = W \Lambda^t W^{-1} w_{l,j} = \sum_{i=1}^{s_l} B_l^t(i, j) w_{l,i}.$$

Combined with the decomposition $\delta_y = \sum_{l,j} \bar{w}_{l,j}(y) \pi(y) w_{l,j}$, this implies that

$$v_t^{x_0}(y) = k^{-t} \sum_{l=1}^m \sum_{i=1}^{s_l} \sum_{j=i}^{s_l} w_{l,i}(x_0) B_l^t(i, j) \bar{w}_{l,j}(y) \pi(y),$$

and plugging in the fact that for $l = 1$ we have $w_{1,1} = \underline{1}$ and $B_1 = (k)$ then yields

$$v_t^{x_0}/\pi = \underline{1} + k^{-t} \sum_{l=2}^m \sum_{i=1}^{s_l} \sum_{j=i}^{s_l} w_{l,i}(x_0) B_l^t(i, j) \bar{w}_{l,j}.$$

Since the $w_{l,i}$'s form an orthonormal basis with respect to $\langle \cdot, \cdot \rangle_{L^2(\pi)}$, we get

$$\begin{aligned} \|v_t^{x_0}/\pi - \underline{1}\|_{L^2(\pi)}^2 &= k^{-2t} \sum_{l=2}^m \sum_{j=1}^{s_l} \left| \sum_{i=1}^j w_{l,i}(x_0) B_l^t(i, j) \right|^2 \\ &\leq k^{-2t} \sum_{l=2}^m \sum_{j=1}^{s_l} j \sum_{i=1}^j |w_{l,i}(x_0)|^2 |B_l^t(i, j)|^2 \leq r^3 b^{r-1} t^{2(r-1)} k^{-t} \sum_{l,i} |w_{l,i}(x_0)|^2, \end{aligned}$$

where the first inequality in the second line is by Cauchy–Schwarz, and the last one uses (3.5) and the fact that $s_l \leq r$ for all l . Finally, by Parseval’s identity, $\sum_{l,i} |w_{l,i}(x_0)|^2$ is equal to $\pi(x_0)^{-2} \|\delta_{x_0}\|_{L^2(\pi)}^2 = \pi(x_0)^{-1}$, and therefore we conclude that

$$\|v_t^{x_0}/\pi - \underline{1}\|_{L^2(\pi)}^2 \leq r^3 b^{r-1} t^{2(r-1)} \pi(x_0)^{-1} k^{-t}. \quad (3.6)$$

Plugging in $t = t_\star$ from (3.3) in (3.6) establishes (3.2), as desired.

It remains to treat the case $M \geq 2$, allowing periodicity. Denote by m the period of $A_{\mathcal{Y}}$ (so that $m \mid M$ by assumption), let $V(\mathcal{Y}) = V_0 \cup \dots \cup V_{m-1}$ be an m -partition of \mathcal{Y} such that all the out-neighbors of a vertex $x \in V_j$ belong to $V_{j+1 \pmod m}$, and let π_j denote the uniform distribution over V_j .

We claim that, analogously to the bound (3.2), we have for all $j = 0, \dots, M-1$,

$$\max_{x_0 \in V_j} \|v_{t_\star+M}^{x_0}/\pi_j - \underline{1}\|_{L^2(\pi_j)}^2 \leq \left(\frac{r^3 b^{r-1}}{(\log k)^{2r-2}} + o(1) \right) \frac{1}{\log n}. \quad (3.7)$$

To see this, recall that by the Perron–Frobenius Theorem for irreducible matrices and our hypothesis on \mathcal{Y} , all eigenvalues of $A_{\mathcal{Y}}$ either lie within the ball of radius \sqrt{k} , or they are of the form $\exp(2\pi i j/m)k$ for $j = 0, \dots, m-1$ (each appearing with multiplicity 1). In particular, as $m \mid M$, the digraph \mathcal{Y}'_j corresponding to the M -th power of \mathcal{Y} restricted to V_j for a given j (that is, $A_{\mathcal{Y}'_j} = A_{\mathcal{Y}}^M|_{V_j}$) has a single multiplicity for the Perron eigenvalue $k' = k^M$ (the out-degree) and all of its other eigenvalues lie within the ball of radius $k^{M/2} = \sqrt{k'}$. Therefore, the inequality (3.2) is valid for the digraph \mathcal{Y}' at t'_\star given by

$$t'_\star = \lceil \log_{k'} n + (2r-1+\alpha) \log_{k'} \log n \rceil = \lceil M^{-1} \log_k n + M^{-1} (2r-1+\alpha) \log_k \log n \rceil.$$

The proof of (3.7) now follows from the fact that t'_\star steps in \mathcal{Y}' correspond to $Mt'_\star \leq t_\star + M$ steps of the random walk in the original digraph \mathcal{Y} , and the L^2 -distance is non-increasing. Consequently, recalling that (Y_t) is the random walk that is modified in its first step to walk according to $A_{\mathcal{Y}}^j$ where j is uniformly chosen over $1, \dots, M$, its distance from the uniform distribution at time $t_\star + M = \log_k n + (2r-1+\alpha+o(1)) \log_k \log n$ is $o(1)$, as claimed. \square

Proof of Theorem 2. By Proposition 3.1, the simple random walk Y_t on the digraph \mathcal{Y} , modified in its first steps to perform $U_0 \sim \text{Uniform}(\{1, \dots, M\})$ steps instead of a single one, satisfies $\max_{x_0} \|\mathbb{P}_{x_0}(Y_{t_\star} \in \cdot) - \pi\|_{\text{TV}} = o(1)$ at t_\star from (3.3) (see (3.2)). In particular, for every initial vertex x_0 , there is a directed path of length at most $t_\star + M$ from x_0 to all but at most $o(n)$ vertices of \mathcal{Y} . Applying the same argument on the reverse digraph—the one whose adjacency matrix is $A_{\mathcal{Y}}^1 = A_{\mathcal{Y}}^*$, which is again k -regular, r -normal and Ramanujan—shows that for every x_0 , there is a directed path of length at most $t_\star + M$ to x_0 from all but $o(n)$ vertices of \mathcal{Y} . Altogether, the diameter of \mathcal{Y} is at most $2(t_\star + M)$. \square

4. Random walks on Ramanujan complexes

In this section we establish Theorem 3, showing that collision-free operators on affine buildings induce r -normal Ramanujan digraphs on their Ramanujan quotients; this will be achieved by Propositions 4.1, 4.3 and 4.5. We will then combine these results with Proposition 3.1 to conclude the proof of Theorem 1.

Recalling the definition of a Ramanujan complex in (1.1), let us mention that for the special case where G is of type \tilde{A}_n the definition here is a priori stronger than the one given in [30, 36]. There the authors used a maximal compact subgroup K — a stabilizer of a vertex — instead of \mathcal{I} . As K contains \mathcal{I} , the definition here may be stronger. But as observed in [16], the explicit examples constructed in [35] are Ramanujan also with this stronger definition, since the work of Lafforgue [28], upon which [35] is based, is valid for L -spherical representations for every compact open subgroup L . In fact, we do not know any example of X which is Ramanujan in one sense and not in the other (and for \tilde{A}_d , $d = 1, 2$, these notions are indeed known to be equivalent—see [24]). Anyway, the stronger definition we use here seems to be the “right” one (see [16] for a thorough discussion) and certainly the one which makes sense for all simple groups G . Note that for all G ’s as above, the Iwahori subgroups are uniquely determined up to conjugacy, but the maximal compact subgroups are not.

This stronger definition allows us to deduce bounds on the norms of essentially all the combinatorial/geometric operators acting on X , not only those defined in terms of its 1-skeleton, as done in [36]. Sometimes, [13, 17] are good examples, the information on the K -spherical representations suffices even for the study of high-dimensional issues, as X is a clique complex which is completely determined by its 1-skeleton. Still, in the current paper we make essential use of operators acting on higher-dimensional cells, and we do need the full power of (1.1).

Proposition 4.1. *If T is a G -equivariant branching operator on $\mathfrak{C} \subset \mathcal{B}$, and $X = \Gamma \backslash \mathcal{B}$ is Ramanujan, then every non-trivial eigenvalue of T acting on $\mathfrak{X} = \Gamma \backslash \mathfrak{C}$ is in the 2^+ -point spectrum of T acting on \mathfrak{C} .*

Remark 4.2. The trivial eigenvalues of T are those arising from one-dimensional representations of G , as explained in the proof. In the case that G is simple (which one can assume, without loss of generality), an eigenvalue is trivial if and only if the corresponding eigenfunction, when lifted from $\Gamma \backslash \mathfrak{C}$ to \mathfrak{C} , is G -invariant.

Proof of Proposition 4.1. Let Σ be a collection of faces of the chamber σ_0 such that $\mathfrak{C} = \coprod_{\sigma \in \Sigma} G\sigma$, and denote by P_σ the G -stabilizer of each $\sigma \in \Sigma$.² For a representation V of G denote

$$V^\Sigma = \bigoplus_{\sigma \in \Sigma} V^{P_\sigma} \leq \bigoplus_{\sigma \in \Sigma} V,$$

² The proof is simpler in the special case where T acts only on the top-dimensional cells, i.e., $\mathfrak{C} = \mathcal{B}(d)$ and $\Sigma = \{\sigma_0\}$; the reader may want to first focus on that case of the proposition.

which is a representation space of the algebra $M_{\Sigma \times \Sigma}(\mathbb{C}G)$. Since $L^2(\Gamma \backslash G\sigma) \cong L^2(\Gamma \backslash G/P_\sigma) \cong L^2(\Gamma \backslash G)^{P_\sigma}$, we can identify $L^2(\mathfrak{X})$ with $L^2(\Gamma \backslash G)^\Sigma$, and in particular for $\Gamma = 1$ we obtain $L^2(\mathfrak{C}) \cong L^2(G)^\Sigma$. Explicitly, every cell in $\mathfrak{X} = \Gamma \backslash \mathfrak{C}$ is of the form $\Gamma g\sigma$ for some $g \in G$ and $\sigma \in \Sigma$, and for $(f_\sigma)_{\sigma \in \Sigma} \in \bigoplus_{\sigma \in \Sigma} L^2(\Gamma \backslash G)^{P_\sigma}$ we have

$$L^2(\mathfrak{X}) \cong L^2(\Gamma \backslash G)^\Sigma \quad \text{via} \quad f(\Gamma g\sigma) = f_\sigma(\Gamma g). \quad (4.1)$$

Turning to the operator T , let $S^{\sigma, \tau}$ be subsets of G , indexed by $\sigma, \tau \in \Sigma$, such that

$$T(\sigma) = \{s\tau \mid \tau \in \Sigma, s \in S^{\sigma, \tau}\}.$$

Since T is G -equivariant, this implies that $T(g\sigma) = \{gs\tau \mid \tau \in \Sigma, s \in S^{\sigma, \tau}\}$ for any $g \in G$ and $\sigma \in \Sigma$, and thus $A_{\mathcal{Y}_{T, \mathfrak{X}}}$ acts on $L^2(\mathfrak{X})$ by

$$(A_{\mathcal{Y}_{T, \mathfrak{X}}} f)(\Gamma g\sigma) = \sum_{\tau \in \Sigma} \sum_{s \in S^{\sigma, \tau}} f(\Gamma gs\tau).$$

Thus, under the identification (4.1), $A_{\mathcal{Y}_{T, \mathfrak{X}}}$ acts as the element $\mathcal{A} \in M_{\Sigma \times \Sigma}(\mathbb{C}G)$ with $(\mathcal{A})_{\sigma, \tau} = \sum_{s \in S^{\sigma, \tau}} s$ (in particular, V^Σ is \mathcal{A} -stable). Since Γ is cocompact, $L^2(\Gamma \backslash G)$ decomposes as an orthogonal Hilbert sum of unitary irreducible representations of G ,

$$L^2(\Gamma \backslash G) = \bigoplus_i V_i \oplus \bigoplus_j W_j \oplus \bigoplus_k U_k,$$

where the V_i are the infinite-dimensional \mathcal{I} -spherical representations which appear in $L^2(\Gamma \backslash G)$, the W_j are the finite-dimensional ones, and the U_k are the ones with no \mathcal{I} -fixed vectors. This implies

$$L^2(\mathfrak{X}) \cong L^2(\Gamma \backslash G)^\Sigma = \bigoplus_i V_i^\Sigma \oplus \bigoplus_j W_j^\Sigma, \quad (4.2)$$

and there are only finitely many summands in this decomposition, as X is finite. Since $A_{\mathcal{Y}_{T, \mathfrak{X}}}$ coincides with an element in $M_{\Sigma \times \Sigma}(\mathbb{C}G)$, it respects the decomposition (4.2) and thus the spectrum of T on \mathfrak{X} is

$$\text{spec}(A_{\mathcal{Y}_{T, \mathfrak{X}}}) = \bigcup_i \text{spec}(\mathcal{A}|_{V_i^\Sigma}) \cup \bigcup_j \text{spec}(\mathcal{A}|_{W_j^\Sigma}).$$

The trivial eigenvalues of T on \mathfrak{X} are by definition the ones in $\bigcup_j \text{spec}(\mathcal{A}|_{W_j^\Sigma})$. When G is simple, W_j is only the trivial representation, but we are also interested in the case where the algebraic group \mathbf{G} is simple, but the group $G = \mathbf{G}(F)$ is not, e.g., $\mathbf{G} = \text{PGL}_{d+1}$. In this case, which is explored in §5.1, there can be trivial eigenvalues arising from non-trivial representations. Every non-trivial eigenvalue λ of T on X is obtained as an eigenvalue of \mathcal{A} acting on some eigenfunction $(f_\sigma)_{\sigma \in \Sigma} \in V_i^\Sigma$. Fix $\xi \in \Sigma$ for which $f_\xi \neq 0$, and for every $\sigma \in \Sigma$ denote by $\varphi_\sigma : G \rightarrow \mathbb{C}$ the matrix coefficient

$$\varphi_\sigma(g) = \langle gf_\sigma, f_\xi \rangle = \int_{\Gamma \backslash G} f_\sigma(\Gamma xg) \overline{f_\xi(\Gamma x)} d(\Gamma x).$$

Since f_σ is P_σ -invariant, so is φ_σ , which means that it defines a function on $G\sigma$. We can patch these functions together to form a function on \mathfrak{C} , namely $\varphi(g\sigma) = \varphi_\sigma(g)$ (for any $g \in G$ and $\sigma \in \Sigma$). The function φ is a λ -eigenfunction of T on \mathfrak{C} since

$$\begin{aligned} (A\mathcal{Y}_{T,\mathfrak{X}}\varphi)(g\sigma) &= \sum_{\tau \in \Sigma} \sum_{s \in S^{\sigma,\tau}} \varphi(gs\tau) = \sum_{\tau \in \Sigma} \sum_{s \in S^{\sigma,\tau}} \varphi_\tau(gs) = \sum_{\tau \in \Sigma} \sum_{s \in S^{\sigma,\tau}} \langle gs f_\tau, f_\xi \rangle \\ &= \left\langle \sum_{\tau \in \Sigma} \mathcal{A}_{\sigma,\tau} f_\tau, g^{-1} f_\xi \right\rangle = \langle \lambda f_\sigma, g^{-1} f_\xi \rangle = \lambda \varphi_\sigma(g) = \lambda \varphi(g\sigma), \end{aligned}$$

and it is not zero since $\varphi(\xi) = \varphi_\xi(e) = \|f_\xi\|^2 \neq 0$.

It remains to show that φ is in $L^{2+\varepsilon}(\mathfrak{C})$ for all $\varepsilon > 0$, and for this we need the Ramanujan assumption (1.1). Fix a good maximal compact subgroup K of G , and let $d_\sigma = \dim \text{span}(K f_\sigma)$. Observe that $d_\sigma \leq [K : P_\sigma] < \infty$ since P_σ is open. Since V_i is \mathcal{I} -spherical (as $\mathcal{I} \leq P_\sigma$ for all σ) and infinite-dimensional, by (1.1) it is weakly contained in the regular representation of G . Thus, by [6, Theorem 2],

$$|\varphi_\sigma(g)| = |\langle g f_\sigma, f_\xi \rangle| \leq \sqrt{d_\sigma d_\xi} \|f_\sigma\| \|f_\xi\| \Xi(g),$$

where Ξ is the Harish-Chandra function, which is in $L^{2+\varepsilon}(G)$ for all $\varepsilon > 0$ (loc. cit.). In particular,

$$\begin{aligned} \|\varphi\|_{L^{2+\varepsilon}(\mathfrak{C})}^{2+\varepsilon} &= \sum_{\sigma \in \Sigma} \sum_{g \in G/P_\sigma} |\varphi_\sigma(g)|^{2+\varepsilon} = \sum_{\sigma \in \Sigma} \frac{1}{\mu(P_\sigma)} \int_G |\varphi_\sigma(g)|^{2+\varepsilon} dg \\ &\leq (\sqrt{d_\sigma d_\xi} \|f_\sigma\| \|f_\xi\| \|\Xi\|_{2+\varepsilon})^{2+\varepsilon} \sum_{\sigma \in \Sigma} \frac{1}{\mu(P_\sigma)} < \infty \end{aligned}$$

for every $\varepsilon > 0$, so that λ is in the 2^+ -point spectrum of T acting on \mathfrak{C} . \square

Before we move on to the trivial eigenvalues, we recall some terminology and facts regarding affine buildings (see [4, 18], for example). Let \mathcal{I} be the stabilizer of a fixed chamber σ_0 in \mathcal{B} , \mathfrak{A} be an apartment which contains σ_0 , and \mathcal{N} the stabilizer of \mathfrak{A} . Let (\mathcal{W}, S) be the Coxeter system corresponding to (\mathfrak{A}, σ_0) , i.e., $\mathcal{W} = \mathcal{N}/(\mathcal{N} \cap \mathcal{I}) = \langle S \rangle$, where S is the set of reflections in the walls of σ_0 . When G is simple, the affine Weyl group \mathcal{W} acts simply transitively on the chambers in \mathfrak{A} . Letting C_g denote the double coset $\mathcal{I}g\mathcal{I}$, G decomposes as a disjoint union $G = \bigsqcup_{w \in \mathcal{W}} C_w$, the so-called Iwahori–Bruhat decomposition. Furthermore, \mathcal{W} decomposes as $\mathcal{W}_{\text{tr}} \rtimes W_G$ where W_G is the (finite) spherical Weyl group of G , and \mathcal{W}_{tr} is the coroot lattice, which consists of all translation elements in \mathcal{W} .

Proposition 4.3. *Let T denote a collision-free, G -equivariant, k -regular branching operator on $\mathfrak{C} \subseteq \mathcal{B}$, let $X = \Gamma \backslash \mathcal{B}$ be a Ramanujan complex, and let $\mathfrak{X} = \Gamma \backslash \mathfrak{C}$. Then the trivial eigenvalues of T on \mathfrak{X} all have modulus k .*

Proof. Recall that the trivial eigenvalues of T on \mathfrak{X} correspond to $\bigcup_j \text{spec}(\mathcal{A}|_{W_j^\Sigma})$, in the notation of the previous proof. Each W_j is a one-dimensional representation of G , so that \mathcal{A} acts on W_j^Σ by a $|\Sigma| \times |\Sigma|$ -matrix. At this point we assume that G is simple, by

restricting it to a finite index subgroup if necessary, and enlarging Σ accordingly. This incurs no loss of generality, as T is equivariant with respect to the smaller group as well. Now, in the decomposition of $L^2(\Gamma \backslash \mathfrak{C})$ in (4.2), each W_j is the trivial representation, thus

$$(\mathcal{A}|_{W_j^\Sigma})_{\sigma, \tau} = \#\{\tau' \in T(\sigma) : \tau' \in G\tau\}.$$

This is precisely the adjacency matrix of $\mathcal{Y}_{T, \Sigma}$, the digraph corresponding to T acting on $\Sigma \simeq G \backslash \mathfrak{C}$ (which can have loops and multiple edges). Thus, the trivial spectrum of T on \mathfrak{X} is the spectrum of T on Σ . We will show that each vertex in $\mathcal{Y}_{T, \Sigma}$ has a unique out-neighbor, repeating with multiplicity k . This implies that $\mathcal{Y}_{T, \Sigma}$ is a disjoint union of cycles, each repeating with multiplicity k . It follows that the eigenvalues of $\mathcal{Y}_{T, \Sigma}$ are all of modulus k .

Assume to the contrary that some $\sigma \in \Sigma$ has two different out-neighbors τ_1, τ_2 in $\mathcal{Y}_{T, \Sigma}$. This means that in \mathfrak{C} the set $T(\sigma)$ contains cells τ'_1, τ'_2 from the distinct orbits $G\tau_1$ and $G\tau_2$. Since $\mathcal{Y}_{T, \Sigma}$ is Eulerian (the in-degree of each vertex equals its out-degree), there exists a (directed) cycle C_1 in $\mathcal{Y}_{T, \Sigma}$ which begins with the edge $\sigma \rightarrow \tau_1$ (and ends at σ). Since $\mathcal{Y}_{T, \Sigma} \setminus E(C_1)$ is still Eulerian, there is also a cycle C_2 which begins with $\sigma \rightarrow \tau_2$. Thus, for $m_i = \text{length}(C_i)$ ($i = 1, 2$) there exist $g_1, g_2 \in G$ such that $g_i\sigma \in T^{m_i-1}(\tau'_i) \subseteq T^{m_i}(\sigma)$. Write $g_i = b_i w_i b'_i$ with $b_i, b'_i \in \mathcal{I}$ and $w_i \in \mathcal{W}$, using the Iwahori–Bruhat decomposition. Since $I \leq P_\sigma$, $b_i w_i \sigma = b_i w_i b'_i \sigma \in T^{m_i-1}(\tau'_i)$, which implies that $w_i \sigma \in T^{m_i-1}(b_i^{-1} \tau'_i)$. Note that $b_1^{-1} \tau'_1 \neq b_2^{-1} \tau'_2$, since they are not in the same G -orbit, and both are descendants of σ since $b_i^{-1} \tau'_i \in T(b_i^{-1} \sigma) = T(\sigma)$. Therefore, we must have $w_1 \sigma \neq w_2 \sigma$ by the collision-free assumption. Taking $m = |W_G|$, we have $w_1^m w_2^m \sigma = w_2^m w_1^m \sigma$, since w_i^m lie in the abelian group \mathcal{W}_{tr} , and we now show that this is a common descendant of $w_1 \sigma$ and $w_2 \sigma$, arriving at a contradiction. Indeed, since $w_i \sigma = b_i^{-1} g_i \sigma \in T^{m_i}(b_i^{-1} \sigma) = T^{m_i}(\sigma)$ we have

$$\begin{aligned} w_1^m w_2^m \sigma &\in w_1^m w_2^{m-1} T^{m_2}(\sigma) = w_1^m w_2^{m-2} T^{m_2}(w_2 \sigma) \subseteq w_1^m w_2^{m-2} T^{2m_2}(\sigma) = \dots \\ &\dots \subseteq w_1^{m_1} T^{mm_2}(\sigma) = \dots = w_1 T^{(m-1)m_1+mm_2}(\sigma) = T^{(m-1)m_1+mm_2}(w_1 \sigma), \end{aligned}$$

and in a similar manner $w_2^m w_1^m \sigma \in T^{(m-1)m_2+mm_1}(w_2 \sigma)$. □

We next wish to show that when T acts on $L^2(\mathfrak{X})$, it is r -normal for some r , depending only on G . To this end, we need the following proposition.

Proposition 4.4 ([5, Prop. 2.6]). *Every unitary irreducible representation V of G with \mathcal{I} -fixed vectors can be embedded in a principal series representation.*

Each principal series representation of G is obtained by induction from a character χ of a minimal parabolic subgroup B (a Borel subgroup):

$$\text{Ind}_B^G \chi = \{f : G \rightarrow \mathbb{C} : f(bg) = \chi(b)f(g) \forall b \in B, g \in G\}.$$

Thus, if $f \in V^{\mathcal{I}} \hookrightarrow (\text{Ind}_B^G \chi)^{\mathcal{I}}$ then $f(bgi) = \chi(b)f(g)$ for every $b \in B, g \in G, i \in \mathcal{I}$, and the decomposition $G = \coprod_{w \in W_G} Bw\mathcal{I}$ (see [5, (19)]) shows that $\dim V^{\mathcal{I}} \leq |W_G|$.

Thus, each summand in the decomposition of $L^2(\mathfrak{X})$ in (4.2) is of dimension at most $|\Sigma| |W_G|$. Since T acts on $L^2(\mathfrak{X})$ by an element in the group algebra of G , it decomposes with respect to (4.2), and therefore it is $(|\Sigma| |W_G|)$ -normal. In particular, since $|\Sigma| \leq 2^{\text{rank } G+1}$ we arrive at the following.

Proposition 4.5. *If T is a G -equivariant branching operator on $\mathfrak{C} \subset \mathcal{B}$, and $X = \Gamma \backslash \mathcal{B}$ is a finite quotient of \mathcal{B} , then $A_{\mathcal{Y}_{T,\mathfrak{X}}}$ is r -normal for some $r = r_G$.*

Proof of Theorem 3. By Proposition 4.1, every non-trivial eigenvalue λ of T on \mathfrak{X} is also in the 2^+ -point spectrum of T on \mathfrak{C} , so $|\lambda| \leq \sqrt{k}$ by Proposition 2.3. The trivial eigenvalues all have modulus k by Proposition 4.3, thus $\mathcal{Y}_{T,\mathfrak{X}}$ is Ramanujan. Finally, Proposition 4.5 shows that $\mathcal{Y}_{T,\mathfrak{X}}$ is r -normal for some $r = r_G$. \square

Proof of Theorem 1. By Theorem 3, the finite digraph $\mathcal{Y} = \mathcal{Y}_{T,\mathfrak{X}}$ is Ramanujan as well as r -normal for some $r = r_G$. It is also k -regular, being the quotient of the k -regular digraph $\mathcal{Y}_{T,\mathfrak{C}}$, and thus its stationary distribution π is uniform. Further note that the hypothesis $\max_{x,y} \rho(x,y) < \infty$ in Theorem 1 says that \mathcal{Y} is strongly connected. Finally, to bound the period of \mathcal{Y} , recall that all of its non-trivial eigenvalues have modulus at most \sqrt{k} (being Ramanujan), and hence its period is bounded by the number of trivial eigenvalues (those with modulus k), which is at most $|G/G'| |\Sigma| \leq |G/G'| 2^{\text{rank } G+1}$, where G' is the derived subgroup of G .

We may thus apply Proposition 3.1 to \mathcal{Y} (e.g., taking $M := (|G/G'| 2^{\text{rank } G+1})!$ to be divisible by its period), establishing part (i) of Theorem 1, whereas part (ii) is an immediate consequence of it as in the proof of Theorem 2 in §3. \square

Remark 4.6. Another way to explain the results of this section is by considering $\mathcal{H} = \mathcal{H}(G, \mathcal{I})$, the Iwahori–Hecke algebra of G . This is the algebra of all complex bi- \mathcal{I} -invariant functions on G with compact support, with respect to convolution. As G acts on $L^2(\Gamma \backslash G)$ (from the right), \mathcal{H} acts on $L^2(\Gamma \backslash G/\mathcal{I})$, the space of complex functions on the chambers of X .

Every irreducible G -subrepresentation U of $L^2(\Gamma \backslash G)$ with $U^{\mathcal{I}} \neq \{0\}$ induces an irreducible representation of the Iwahori–Hecke algebra \mathcal{H} on $U^{\mathcal{I}}$ which is a subspace of $L^2(\Gamma \backslash G)^{\mathcal{I}} = L^2(\Gamma \backslash G/\mathcal{I}) = L^2(X(d))$, where $X(d)$ is the set of d -dimensional cells of $X = \Gamma \backslash \mathcal{B}$. Condition (1.1) enables us to deduce that for operators $T \in \mathcal{H}$, the non-trivial eigenvalues of T acting on X appear in the spectrum of T acting on $L^2(\mathcal{B}(d))$. The irreducible representations of \mathcal{H} are all of bounded dimensions, which ensures that T decomposes (with respect to a suitable orthonormal basis) as a sum of blocks of bounded size. For a thorough treatment of this approach, we refer the reader to [16, 21].

5. Collision-free operators on affine buildings

In this section we study collision-free branching operators on buildings. In §5.1 we focus on the buildings associated with PGL_m , and explore a family of operators which originate in the works of Li and Kang [22, 23], and can be thought of as non-Archimedean geodesic flows. In §5.2, we give, for a general simple group G , a necessary and sufficient condition for a G -equivariant branching operator on the chambers of the building of G , to be collision-free.

5.1. Geodesic flows on complexes of type \tilde{A}_d

Let F be a non-Archimedean local field with ring of integers \mathcal{O} , uniformizer π , and residue field $\mathcal{O}/\pi\mathcal{O}$ of order q . Let $G = \mathrm{PGL}_m(F)$, and fix the maximal compact subgroup $K = \mathrm{PGL}_m(\mathcal{O})$. The building $\mathcal{B} = \mathcal{B}(\mathrm{PGL}_m(F))$ is a simplicial complex of dimension $d = m - 1$, whose vertices correspond to the G/K -cosets, and they are “colored” by the function

$$\mathrm{col} : G/K \rightarrow \mathbb{Z}/m\mathbb{Z}, \quad \mathrm{col}(gK) \equiv \mathrm{ord}_\pi(\det(g)) \pmod{m}.$$

Each $g \in G$ corresponds to a homothety class of \mathcal{O} -lattices in F^m , namely $g\mathcal{O}^m$. The vertices g_0K, \dots, g_jK form a cell in \mathcal{B} if and only if each homothety class $g_i\mathcal{O}^m$ can be represented by a lattice L_i , so that, possibly after reordering,

$$L_0 > L_1 > \dots > L_j > \pi L_0.$$

The edges of \mathcal{B} are also colored: if $e = (x, y)$, then $\mathrm{col}(e) = \mathrm{col}(y) - \mathrm{col}(x)$ (as an element of the group $\mathbb{Z}/m\mathbb{Z}$). Note that while the color of vertices is not G -invariant, the color of edges is.

The geodesic flow on \mathcal{B} which we now define is a simplicial analog of the geodesic flow on the unit bundle of a Riemannian manifold, where the role of the “direction” vector is played by a cell. Formally, for every $1 \leq j \leq m - 1$, we define the “ j -th-unit bundle” $UT^j\mathcal{B}$ as the set of pairs $(v \in \mathcal{B}(0), \sigma \in \mathcal{B}(j))$ (where $\mathcal{B}(j)$ is the set of j -dimensional cells) such that $v \in \sigma$, and the vertices in σ are of colors $\mathrm{col}(v), \mathrm{col}(v) + 1, \dots, \mathrm{col}(v) + j$. In the next step of the flow, the basepoint v is replaced by the “next” vertex in σ (the one with color $\mathrm{col}(v) + 1$). A more difficult question is what happens to the “direction” σ , as there is no notion of parallel transport in discrete settings. In fact, there are several possible options, which is why we obtain a branching operator, and not a deterministic flow.

Definition 5.1. The geodesic flow T on $UT^j\mathcal{B}$ is defined as follows: $T(\sigma)$ for $\sigma = (v_0, \{v_0, \dots, v_j\}) \in UT^j\mathcal{B}$, with $\mathrm{col}(v_i) \equiv \mathrm{col}(v_0) + i$, consists of all the cells $\sigma' = (v_1, \{v_1, \dots, v_j, w\}) \in UT^j\mathcal{B}$ such that

- (1) $\mathrm{col}(w) \equiv \mathrm{col}(v_1) + j$ (the “direction vector” is based at v_1),
- (2) $\{v_0, v_1, \dots, v_j, w\}$ is not a cell (geodicty).

Note that T is G -equivariant, so it defines a branching operator on any quotient $X = \Gamma \backslash \mathcal{B}$. For example, in dimension one, the geodesic flow on UT^1X coincides with the non-backtracking walk on the graph X , since any two neighbors v, w satisfy $\mathrm{col}(w) \equiv \mathrm{col}(v) + 1$, and there are no triangles, so (2) reduces to $w \neq v_0$. When X is of higher dimension, (2) also prevents the flow on UT^1X from making two steps along the edges of a triangle, as there is a “shorter route”.

Let us analyze the geodesic flow in the language of Section 4. Denote by $v^{(j)}$ the vertex

$$v^{(j)} = \mathrm{diag}(1^{\times j}, \pi^{\times(m-j)}) \cdot K,$$

and by $\sigma^{(j)}$ the ordered cell

$$\sigma^{(j)} = [v^{(j)}, v^{(j-1)} \dots, v^{(1)}, v^{(0)}].$$

Since G acts transitively on $UT^j\mathcal{B}$, and the stabilizer of a pair (v, σ) is the pointwise stabilizer of σ , we can identify the pair (v, σ) with the unique ordering $[v_0, \dots, v_j]$ of σ with consecutive colors and $v_0 = v$. Furthermore, if P_j is the pointwise stabilizer of $\sigma^{(j)}$, then $gP_j \mapsto (gv^{(j)}, g\sigma^{(j)})$ identifies G/P_j with $UT^j\mathcal{B}$. Explicit computation gives

$$\begin{aligned} P_j &= \bigcap_{i=0}^j \text{diag}(1^{\times i}, \pi^{\times(m-i)}) \cdot K \cdot \text{diag}(1^{\times i}, (1/\pi)^{\times(m-i)}) \\ &= \{g \in K : g_{r,c} \in \pi\mathcal{O} \text{ for } c \leq \min(j, r-1)\}. \end{aligned} \quad (5.1)$$

As $v^{(0)}$ corresponds to \mathcal{O}^m , the cells in $T(\sigma^{(j)})$ are of the form $[v^{(j-1)} \dots, v^{(1)}, v^{(0)}, w]$ with w corresponding to a maximal sublattice of \mathcal{O}^m which

- (1) contains $\text{diag}(1^{\times(j-1)}, \pi^{\times(m-j+1)}) \cdot \mathcal{O}^m$, so that $[v^{(j-1)}, \dots, v^{(0)}, w]$ is a cell,
- (2) does not contain $\text{diag}(1^{\times j}, \pi^{\times(m-j)}) \cdot \mathcal{O}^m$, so that the flow is geodesic.

The sublattices which satisfy these two conditions are those of the form

$$L_{a_1, \dots, a_{m-j}} = \left(\begin{array}{c|c|c} I_{j-1} & & \\ \hline & \pi & a_1 \cdots a_{m-j} \\ \hline & & I_{m-j} \end{array} \right) \cdot \mathcal{O}^m \quad (a_i \in \mathcal{O}/\pi\mathcal{O} \cong \mathbb{F}_q),$$

which in particular shows that T is q^{m-j} -out-regular. As G acts transitively on $UT^j\mathcal{B}$, it is in fact q^{m-j} -regular. To see that it is also collision-free, observe that the matrix

$$g_{a_1, \dots, a_{m-j}} = \left(\begin{array}{c|c|c} & I_{j-1} & \\ \hline \pi & & a_1 \cdots a_{m-j} \\ \hline & & I_{m-j} \end{array} \right)$$

takes $v^{(i)}$ to $v^{(i-1)}$ for $1 \leq i \leq j$, and $v^{(0)}$ to the vertex corresponding to $L_{a_1, \dots, a_{m-j}}$. Together with (5.1) this implies that $P_j g_{0, \dots, 0} \sigma^{(j)} = \{g_{a_1, \dots, a_{m-j}} \sigma^{(j)} : a_i \in \mathbb{F}_q\} = T(\sigma^{(j)})$, and it is now an easy exercise to verify that for any $\ell \in \mathbb{N}$,

$$(P_j g_{0, \dots, 0})^{j\ell} \sigma^{(j)} = \left\{ \left(\begin{array}{c|c|c} \pi^\ell & a_{1,1} \cdots a_{1,m-j} \\ & \vdots & \vdots \\ & \pi^\ell & a_{j,1} \cdots a_{j,m-j} \\ \hline & & I_{m-j} \end{array} \right) \sigma^{(j)} : a_{x,y} \in \mathcal{O}/\pi^\ell \mathcal{O} \right\}.$$

Since these matrices carry \mathcal{O}^m to non-equivalent lattices, the last vertices in $T^{\ell j}(g\sigma^{(j)}) = g(P_j g_{0,\dots,0})^{\ell j} \sigma^{(j)}$ are all different, implying that T^j is collision-free, and hence so is T . We have thus shown:

Proposition 5.2. *The j -dimensional geodesic flow on $\mathcal{B}(\mathrm{PGL}_m)$ is collision-free.*

For the geodesic flow, we can give an explicit bound for the normality of the adjacency operator, and for $j = 1$ we show below that it is optimal.

Proposition 5.3. *The j -dimensional geodesic flow on finite quotients of $\mathcal{B}(\mathrm{PGL}_m)$ is $(m)_j$ -normal, where $(m)_j = m!/(m-j)!$.*

Proof. Let V be an irreducible representation of $G = \mathrm{PGL}_m(F)$ with $V^{P_j} \neq 0$. Let B be the standard Borel, consisting of all upper triangular matrices in G . By [5, Prop. 2.6] there exists a character χ of B which is trivial on $B \cap K$ and

$$V^{P_j} \hookrightarrow (\mathrm{Ind}_B^G \chi)^{P_j} = \{f : G \rightarrow \mathbb{C} : f(bgp) = \chi(b)f(g) \forall b \in B, p \in P_j\}. \quad (5.2)$$

We will show that $\dim(\mathrm{Ind}_B^G \chi)^{P_j} = (m)_j$. Denote by W the Weyl group of G , which consists of all permutation matrices, and let $W_j = W \cap P_j$. Observe that (5.1) implies that $W_j = \mathrm{Sym}_{\{j+1, \dots, m\}}$. If $w_1, \dots, w_{(m)_j}$ is a transversal for W/W_j , we claim that $G = \coprod_{i=1}^{(m)_j} Bw_i P_j$. Indeed, the decomposition $G = \coprod_{w \in W} Bw\mathcal{I}$ already implies that $G = \bigcup_i Bw_i P_j$, and we need to show that the summands are disjoint. If $w_i = bw_k p$ for some $b \in B, p \in P_j$, then $b^{-1} = w_k p w_i^{-1} \in B \cap w_k P w_i^{-1} \subseteq B \cap K$, which implies that the diagonal entries of b^{-1} are in \mathcal{O}^\times . On the other hand, let $r > m - j$, and observe that $p_{r,c} = (w_k^{-1} b^{-1} w_i)_{r,c} = (b^{-1})_{w_k(r), w_i(c)}$, so that

$$p_{w_k^{-1}(w_i(c)), c} = (b^{-1})_{w_i(c), w_i(c)} \in \mathcal{O}^\times.$$

By (5.1), this implies $w_k^{-1}(w_i(c)) \leq c$ for $1 \leq c \leq j$, and thus $w_i(c) = w_k(c)$ for $1 \leq c \leq j$, hence $w_i W_j = w_k W_j$. This implies $i = k$, and thus $G = \coprod_{i=1}^{(m)_j} Bw_i P_j$ as claimed. In addition, since $B \cap w_i P_j w_i^{-1} \subseteq B \cap K$ and $\chi|_{B \cap K} \equiv 1$, the functions

$$f_{i'}(bw_i p) = \chi(b)\delta_{i,i'} \quad (b \in B, i, i' \in \{1, \dots, (m)_j\}, p \in P_j)$$

are well defined, and thus form a basis for $(\mathrm{Ind}_B^G \chi)^{P_j}$. \square

Proposition 5.4. *The one-dimensional geodesic flow on Ramanujan quotients of $\mathcal{B}(\mathrm{PGL}_m)$ is not r -normal for any $r < m$.*

Proof. Let $z_1, \dots, z_m \in \mathbb{C}$ with $|z_i| = 1$ for all i and $\prod_{i=1}^m z_i = 1$, and let

$$\chi : B \rightarrow \mathbb{C}^\times, \quad \chi((b_{ij})) = \prod_{i=1}^m (q^{i-(m+1)/2} z_i)^{\mathrm{ord}_\pi(b_{ii})}.$$

The representation $\mathrm{Ind}_B^G \chi$ is irreducible, unitary and K -spherical. Furthermore, if $X = \Gamma \backslash \mathcal{B}$ is Ramanujan, then every K -spherical infinite-dimensional subrepresentation of $L^2(\Gamma \backslash G)$ is of this form (see [36]). Since $L^2(X(0)) \cong L^2(\Gamma \backslash G)^K$ and

$\dim(\text{Ind}_B^G \chi)^K = 1$, the number of such representations which appear in $L^2(\Gamma \backslash G)$ is the number of vertices in $\Gamma \backslash B$ minus the number of one-dimensional subrepresentations of $L^2(\Gamma \backslash G)$, which is bounded by m . By the proof of Proposition 5.3, $(\text{Ind}_B^G \chi)^{P_1}$ is m -dimensional, with basis f_1, \dots, f_m defined by

$$f_i(bwp) = \chi(b)\delta_{w(1),i} \quad (b \in B, w \in S_m, p \in P_1).$$

This is an orthogonal basis, albeit not orthonormal—one can verify that $\|f_i\| = q^{i/2}$. Recall that the geodesic flow acts by $\mathcal{A} = \sum_{\vec{a} \in \mathbb{F}_q^{m-1}} g_{\vec{a}}$, where $g_{\vec{a}}$ is the identity matrix with the first row replaced by $(\pi, a_1, \dots, a_{m-1})$. Since $g_{\vec{a}} \in B$,

$$(\mathcal{A}f_i)(e) = \sum_{\vec{a} \in \mathbb{F}_q^{m-1}} f_i(g_{\vec{a}}) = \sum_{\vec{a} \in \mathbb{F}_q^{m-1}} \chi(g_{\vec{a}}) = q^{m-1} \cdot q^{1-(m+1)/2} z_1 \delta_{1,i} = \delta_{1,i} q^{(m-1)/2} z_1.$$

Assume from now on $1 < j \leq m$, and observe $(\mathcal{A}f_i)((1\ j)) = \sum_{\vec{a} \in \mathbb{F}_q^{m-1}} f_i((1\ j)g_{\vec{a}})$. If $a_1 = \dots = a_{j-1} = 0$, and π_j is the identity matrix with the (j, j) -th entry replaced by π , then $u := \pi_j^{-1}(1\ j)g_{\vec{a}}(1\ j) \in U$, the unipotent upper triangular matrices in G . Since $\chi|_U \equiv 1$,

$$\sum_{\vec{a} \in \mathbb{F}_q^{m-1}, a_1=\dots=a_{j-1}=0} f_i((1\ j)g_{\vec{a}}) = \sum_{\dots} f_i(\pi_j u(1\ j)) = q^{m-j} \chi(\pi_j) \delta_{j,i} = \delta_{j,i} q^{(m-1)/2} z_j.$$

Next, if $a_{j-1} \neq 0$, let $\vec{b} = (-\frac{a_1}{a_{j-1}}, \dots, -\frac{a_{j-2}}{a_{j-1}}, \frac{1}{a_{j-1}}, -\frac{a_j}{a_{j-1}}, \dots, -\frac{a_{m-1}}{a_{j-1}})$, and note that $g_{\vec{b}}^{-1}(1\ j)g_{\vec{a}} \in P_1$, so that

$$\sum_{\vec{a} \in \mathbb{F}_q^{m-1}, a_{j-1} \neq 0} f_i((1\ j)g_{\vec{a}}) = (q-1)q^{m-2} \chi(g_{\vec{b}}) \delta_{1,i} = \delta_{1,i} (q-1)q^{(m-3)/2} z_1.$$

Finally, if $a_{j-1} = 0$, let $k = \min\{i : a_{i-1} \neq 0\}$ (note $k < j$), and let b be the identity matrix with the k -th row replaced by

$$b_{k,c} = \begin{cases} 0, & c < k, \\ \pi, & c = k, \\ 1/a_{k-1}, & c = j, \\ -a_{c-1}/a_{k-1}, & \text{otherwise.} \end{cases}$$

Observe that $p := (1\ k)b^{-1}(1\ j)g_{\vec{a}} \in P_1$, so that

$$\sum_{\vec{a} \in S} f_i((1\ j)g_{\vec{a}}) = \sum_{\vec{a} \in S} f_i((1\ k)bp) = (q-1)q^{m-k-1} \chi((1\ k)b) = (q-1)q^{(m-3)/2} z_k \delta_{k,i}$$

for $S = \{\vec{a} \in \mathbb{F}_q^{m-1} : a_j = a_1 = \dots = a_{k-1} = 0, a_k \neq 0\}$. Altogether, we find that \mathcal{A}

acts on $(\text{Ind}_B^G \chi)^{P_1}$, with respect to the orthonormal basis $\{q^{-i/2} f_i\}$, by

$$\begin{pmatrix} q^{(m-1)/2} z_1 & & & & \\ (q-1)q^{(m-2)/2} z_1 & q^{(m-1)/2} z_2 & & & \\ (q-1)q^{(m-1)/2} z_1 & (q-1)q^{(m-2)/2} z_2 & q^{(m-1)/2} z_3 & & \\ \vdots & \vdots & \ddots & \ddots & \\ (q-1)q^{m-2} z_1 & (q-1)q^{(2m-5)/2} z_2 & \dots & (q-1)q^{(m-2)/2} z_{m-1} & q^{(m-1)/2} z_m \end{pmatrix}. \quad (5.3)$$

Each infinite-dimensional K -spherical representation in $L^2(\Gamma \backslash G)$ contributes a block of this form to $\mathcal{A}_{Y_{T,x}}$, and hence, by the uniqueness of the QR -decomposition, it is not r -normal for any $r < m$. \square

Let us remark that for $m = 2$, Ramanujan quotients of the building \mathcal{B} are just Ramanujan graphs, and the one-dimensional geodesic flow is the non-backtracking random walk. In this case, (5.3) coincides with the unitary block decomposition carried out for this walk in [31].

5.2. Branching operators on chambers for simple groups

In this subsection we take $G = \mathbf{G}(F)$ to be any simple group, and determine which G -equivariant branching operators on the chambers of $\mathcal{B} = \mathcal{B}(G)$ are collision-free.

Any G -equivariant branching operator is of the form $T_{\mathcal{G}}(\sigma_0) = \mathcal{I}\mathcal{G}\sigma_0$ for some finite set $\mathcal{G} \subseteq G$. Observe that $T_{\mathcal{G}}(\sigma_0) = T_{\mathcal{I}\mathcal{G}\mathcal{I}}(\sigma_0)$, so that decomposing $\mathcal{I}\mathcal{G}\mathcal{I} = \bigcup_{w \in \mathcal{W}} C_w$ with $\mathcal{W} \subseteq \mathcal{W}$, we see that $T_{\mathcal{G}}$ equals $T_{\mathcal{W}}$.

Proposition 5.5. *If $T = T_{\mathcal{W}}$ is collision-free for $\mathcal{W} \subseteq \mathcal{W}$, then $|\mathcal{W}| = 1$.*

Proof. Assume that \mathcal{W} contains $w_0 \neq w_1$. Denoting $\mu = |W_G|$, one has $w_0^\mu w_1^\mu = w_1^\mu w_0^\mu$ as in the proof of Proposition 4.3. Therefore, $w_0^\mu w_1^\mu(\sigma_0)$ is contained both in $T^{2\mu-1}(w_0\sigma_0) = (\mathcal{I}\mathcal{W}\mathcal{I})^{2\mu-1}w_0\sigma_0$, and in $T^{2\mu-1}(w_1\sigma_0)$. Since $w_0\sigma_0, w_1\sigma_0 \in T(\sigma_0)$, this implies that T is not collision-free. \square

Thus, we can restrict our attention to the case where $T = T_{w_0}$ for a single $w_0 \in \mathcal{W}$. We need the notion of the Weyl length $\ell(w)$ of $w \in \mathcal{W} = \langle S \rangle$, which is the shortest length of a word in S which equals w . In addition, we need the projection $\rho : \mathcal{B} \rightarrow \mathfrak{A}$ defined by the Iwahori–Bruhat decomposition, namely $\rho|_{C_w\sigma_0} \equiv w\sigma_0$ ($\forall w \in \mathcal{W}$). Denote $k = |T_{w_0}(\sigma_0)|$, and let $\mu \geq 1$ be an integer such that $w_0^\mu \in \mathcal{W}_{\text{tr}}$ (for example, we can always take $\mu = |W_G|$).

Theorem 5.6. *With the above notation, the following are equivalent:*

- (1) *The operator T_{w_0} is collision-free.*
- (2) *$|T_{w_0}^\mu(\sigma_0)| = k^\mu$.*
- (3) *$\ell(w_0^\mu) = \mu \cdot \ell(w_0)$.*
- (4) *The random walk defined by T_{w_0} projects onto a deterministic walk in \mathfrak{A} , namely $|\rho(T_{w_0}^j(\sigma_0))| = 1$ for all j .*

Note that if $w_0 \in \mathcal{W}_{\text{tr}}$ then (3) holds with $\mu = 1$, so T_{w_0} is collision-free.

Proof. It is clear that (1) \Rightarrow (2), and we proceed to show that for every $j \in \mathbb{N}$,

$$|T_{w_0}^j(\sigma_0)| = k^j \iff \ell(w_0^j) = j \cdot \ell(w_0) \iff |\rho(T_{w_0}^j(\sigma_0))| = 1, \quad (5.4)$$

from which (2) \Leftrightarrow (3) \Leftarrow (4) would follow.

If one defines $q_s = |C_s \sigma_0|$ for $s \in S$, then whenever $w = s_1 \cdots s_{\ell(w)}$ is a reduced word, the fiber $C_w \sigma_0$ of ρ over $w \sigma_0$ is of size $q_{s_1} \cdots q_{s_{\ell(w)}}$ (see [18, §6.2]). In particular, writing $w_0 = s_1 \cdots s_{\ell(w_0)}$, we have $k = q_{s_1} \cdots q_{s_{\ell(w_0)}}$. For each $1 \leq i \leq \ell(w_0)$, define a branching operator T_i on $\mathcal{B}(d)$ by $T_i(g \sigma_0) = g C_{s_i} \sigma_0$, and observe that $T_{w_0} = T_{\ell(w_0)} \circ \cdots \circ T_1$. Since each T_i is q_{s_i} -regular, $T_{w_0}^j(\sigma_0)$ is of size k^j if and only if

$$\Upsilon := |T_i(T_{i-1}(\cdots(T_1(T_{w_0}^r(\sigma_0)))))| = k^r q_{s_1} \cdots q_{s_i} \quad (5.5)$$

for every $1 \leq r < j$ and $1 \leq i \leq \ell(w_0)$. Recall the Bruhat relations:

$$C_w C_s = \begin{cases} C_{ws}, & \ell(ws) = \ell(w) + 1, \\ C_{ws} \cup C_w, & \ell(ws) = \ell(w) - 1, \end{cases} \quad (5.6)$$

which hold for every $w \in \mathcal{W}$ and $s \in S$. If $\ell(w_0^j) = j \cdot \ell(w_0)$ then $(s_1 \cdots s_{\ell(w_0)})^j$ is a reduced word for w_0^j , so that (5.6) implies

$$T_{w_0}^j(\sigma_0) = (C_{w_0})^j \sigma_0 = (C_{s_1} \cdots C_{s_{\ell(w_0)}})^j \sigma_0 = C_{w_0^j} \sigma_0,$$

from which one infers that $\rho(T_{w_0}^j(\sigma_0)) = \{w_0^j \sigma_0\}$ and $|T_{w_0}^j(\sigma_0)| = k^j$.

In the other direction, if $\ell(w_0^j) < j \cdot \ell(w_0)$, consider the first r, i for which $w_0^r s_1 \cdots s_i$ is not reduced, and let $w' = w_0^r s_1 \cdots s_{i-1}$. By the exchange property of Coxeter groups, one can remove some letter s from w' , obtaining a word w'' such that $w'' s_i$ is a reduced word for w' . This also implies that w'' is a reduced word for $w' s_i$, so that from (5.6) one has

$$C_{w_0^r s_1 \cdots s_{i-1}} C_{s_i} \sigma_0 = C_{w''} \sigma_0 \cup C_{w'' s_i} \sigma_0$$

(a disjoint union, as \mathcal{W} acts freely on \mathfrak{A}). From this one sees that (5.5) fails:

$$\begin{aligned} \Upsilon &= |C_{w_0^r s_1 \cdots s_{i-1}} C_{s_i} \sigma_0| = |C_{w''} \sigma_0| + |C_{w'' s_i} \sigma_0| = \frac{k^r q_{s_1} \cdots q_{s_{i-1}}}{q_s} + \frac{k^r q_{s_1} \cdots q_{s_i}}{q_s} \\ &= k^r q_{s_1} \cdots q_{s_{i-1}} \frac{1 + q_{s_i}}{q_s} < k^r q_{s_1} \cdots q_{s_i}, \end{aligned}$$

since the building associated with a simple group is thick ($q_s \geq 2$). This implies that $|T_{w_0}^{r+1}(\sigma_0)| \leq k^{r+1}$ and $|T_{w_0}^j(\sigma_0)| < k^j$. Furthermore, $|\rho(T_{w_0}^j(\sigma_0))| \geq 2$, since

$$T_{w_0}^j(\sigma_0) = C_{w_0^r s_1 \cdots s_{i-1}} C_{s_i} C_{s_{i+1}} \cdots C_{s_{\ell(w_0)}} (C_{s_1} \cdots C_{s_{\ell(w_0)}})^{j-r-1}(\sigma_0)$$

contains the cells $w'' s_{i+1} \cdots s_{\ell(w_0)} w_0^{j-r-1} \sigma_0$ and $w'' s_i \cdots s_{\ell(w_0)} w_0^{j-r-1} \sigma_0$ (which are different since \mathcal{W} acts simply transitively on the chambers of \mathfrak{A}), yielding (5.4).

Next, we recall the fact (see, e.g., [18, §12.2]) that $\ell(w)$ (for any $w \in \mathcal{W}$) equals the number of walls in \mathfrak{A} separating σ_0 from $w\sigma_0$. In particular, this applies to $w = w_0^\mu$, which acts on \mathfrak{A} by translation, hence for every $j \in \mathbb{N}$ there are $j \cdot \ell(w_0^\mu)$ walls separating σ_0 from $w_0^{\mu j} \sigma_0$. If (3) holds, then $\ell(w_0^{\mu j}) = j \cdot \ell(w_0^\mu) = j\mu\ell(w_0)$. But this means that $(s_1 \cdots s_{\ell(w_0)})^{j\mu}$ is reduced for every j , hence also $\ell(w_0^j) = j\ell(w_0)$ for every j , thus by (5.4) we have (3) \Rightarrow (4). Finally, it is now clear that (2), (3), (4) imply (1): for $j \neq j'$, (4) implies that $T_{w_0}^j(\sigma_0) \cap T_{w_0}^{j'}(\sigma_0) = \emptyset$, and for each j , (4) and (5.4) together imply that $|T_{w_0}^j(\sigma_0)| = k^j$. \square

We remark that we can apply this theorem to the d -dimensional geodesic flow T on PGL_{d+1} , even though it is a non-simple group; one can look at T^{d+1} , which belongs to the simple group PSL_{d+1} and satisfies the conditions of the theorem.

6. Zeta functions of digraphs and Ramanujan complexes

Recall that Ihara [20] associated with every graph \mathcal{G} a zeta function defined by $Z_{\mathcal{G}}(u) = \prod_{[\gamma]} \frac{1}{1-u^{\ell(\gamma)}}$, where $[\gamma]$ runs over the equivalence classes of geodesic tailless primitive cycles γ in \mathcal{G} (two cycles are equivalent if they differ by a cyclic rotation), and $\ell(\gamma)$ is the length of γ . Taking a logarithmic derivative, one gets

$$Z_{\mathcal{G}}(u) = \exp\left(\sum_{m=1}^{\infty} \frac{N_m(\mathcal{G})}{m} u^m\right),$$

where $N_m(\mathcal{G})$ is the number of geodesic tailless cycles in \mathcal{G} of length m . For a k -regular graph \mathcal{G} with $k = q + 1$, Ihara showed that

$$Z_{\mathcal{G}}(u) = [(1 - u^2)^{\beta_1} \det(I - uA_{\mathcal{G}} + u^2qI)]^{-1}, \quad (6.1)$$

where β_1 is the first Betti number of \mathcal{G} . From (6.1) one deduces (see [33, p. 59]) that \mathcal{G} is Ramanujan iff the associated zeta function satisfies the Riemann Hypothesis:

$$Z_{\mathcal{G}}(q^{-s}) = \infty \implies \text{either } q^{-s} \in \{\pm 1, \pm 1/q\} \text{ or } \Re(s) = 1/2,$$

i.e., all the poles of $Z_{\mathcal{G}}(u)$ are obtained at $u = \pm 1, \pm 1/q$, or for $|u| = \sqrt{q}$.

Hashimoto [19] formulated a variant of this by showing

$$Z_{\mathcal{G}}(u) = \det(I - uT)^{-1},$$

where T is the non-backtracking operator on directed edges (coinciding with the case $m = 2, j = 1$ of our geodesic flow in §5.1). This approach generalizes to arbitrary digraphs: indeed, defining the zeta function of a finite digraph \mathcal{Y} to be

$$Z_{\mathcal{Y}}(u) = \prod_{[\gamma]} \frac{1}{1 - u^{\ell(\gamma)}}, \quad (6.2)$$

where $[\gamma]$ runs over the equivalence classes of primitive (directed) cycles in \mathcal{Y} , one has:

Theorem 6.1 ([3, 27]). *If $N_m(\mathcal{Y})$ is the number of (directed) cycles of length m in \mathcal{Y} , then*

$$Z_{\mathcal{Y}}(u) = \exp\left(\sum_{m=1}^{\infty} \frac{N_m(\mathcal{Y})}{m} u^m\right) = \frac{1}{\det(I - uA_{\mathcal{Y}})}.$$

Various authors have suggested various zeta functions associated to simplicial complexes with the hope to generalize the Hashimoto work [7, 15, 22–25]. We will associate a zeta function of this kind with every G -equivariant branching operator:

Definition 6.2. For every G -equivariant branching operator T on $\mathfrak{C} \subseteq \mathcal{B} = \mathcal{B}(G)$, the T -zeta function of a quotient complex $X = \Gamma \backslash \mathcal{B}$ (as in §4) is defined as $z_T(X, u) = Z_{\mathcal{Y}_{T, \mathfrak{X}}}(u)$, namely (6.2), with γ running over the equivalence classes of primitive cycles in the digraph defined by T on \mathfrak{X} .

Theorem 6.1 implies that, for $\mathcal{Y} = \mathcal{Y}_{T, \mathfrak{X}}$, one has

$$z_T(X, u) = \det(I - uA_{\mathcal{Y}})^{-1}.$$

We have shown in §4 that if $X = \Gamma \backslash \mathcal{B}$ is a Ramanujan complex and T is a G -equivariant, k -regular, collision-free branching operator on \mathcal{B} , then $\mathcal{Y}_{T, \mathfrak{X}}$ is Ramanujan. In particular, the T -zeta function of X satisfies the R.H.: if $z_T(X, k^{-s}) = \infty$ then either $|s| = 1$ or $\Re(s) \leq 1/2$, establishing Corollary 4.

Remark 6.3. The bound $|\Re(s)| \leq 1/2$ is the true situation: there are poles with $|\Re(s)| < 1/2$. In fact, already in the graph case there are poles with $|\Re(s)| = 0$, and in higher dimension there are also poles with $0 < |\Re(s)| < 1/2$ (see, e.g., [24]).

Remark 6.4. Zeta functions of this kind have been studied by a number of authors [7, 15, 22–25]. The most general results regarding the R.H. are due to Kang [22], who studied the case of buildings of type \tilde{A}_n and gave more detailed information about the poles of the zeta functions corresponding to geodesic flows, via case by case analysis of the representations of $\mathrm{GL}_n(F)$. In [15], the same is done for the building of type \tilde{C}_2 associated with $\mathrm{PSp}(4)$. Our simple combinatorial treatment gives the upper bound on $\Re(s)$ but not the exact possible values of it. However, our method applies for any building, and not only those of type \tilde{A}_n or \tilde{C}_2 . It seems, in any case, that for combinatorial applications the upper bound on $\Re(s)$ suffices, as is illustrated in this paper.

Remark 6.5. For k -regular graphs, it is known that the converse also holds: the R.H. for the zeta function of a graph implies it is Ramanujan. In [24] it was shown that the R.H. for zeta functions of geodesic flows on a building of type \tilde{A}_2 implies that a quotient complex is Ramanujan. Recently, an analogous result was established by Kamber for buildings of general type [21].

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